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Free-Response Simulation via the Proper Orthogonal Decomposition

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The proper orthogonal decomposition is applied to the response of linear and nonlinear beam models to construct a reduced-order model for predicting the response to various initial conditions without requiring the equations of motion for the structure. The proper orthogonal decomposition is interpreted as a modal sum, and the structural response to an initial condition is expressed as a weighted summation of proper orthogonal modes and corresponding proper orthogonal coordinate histories. A method for calculating the weights of each mode in the response to an altered initial condition is presented. This method is applied to predict the responses of a linear undamped beam model and a damped beam model with a nonlinear spring to various initial displacement and velocity profiles. The results obtained are compared with those from respective finite element models for each beam.

Nomenclature

$\mathbf{u}_i$ = ith proper orthogonal mode
$\mathbf{v}_i$ = ith proper orthogonal coordinate history
W = displacement snapshot matrix
$\mathbf{w}_i$ = ith initial displacement profile
$\dot{\mathbf{w}}_i$ = ith initial velocity profile
ε_i = energy captured by ith proper orthogonal mode
σ_i = ith proper orthogonal value

Introduction

THE finite element (FE) method is commonly used to analyze the dynamics of complex structures [1]. Although the method is very powerful, some of its limitations become apparent when it is applied for analysis of very large, possibly nonlinear, structures with millions of degrees of freedom. Months may be required to develop the geometry and form the element mesh for such models. After the model is completed, the analysis may require days or weeks of processing time [2]. Finally, there is no guarantee that the analysis will accurately predict the behavior of an actual structure. The analysis may be incorrect due to modeling errors (e.g., incorrect assumptions about damping or linearity), parameter errors (e.g., inaccuracy of Young's modulus), or other factors [3].

Many methods have been developed to circumvent these problems by using experimental response data to characterize or identify a system. Modal analysis is commonly used to validate or update linear FE models to insure that the computer analyses will be correct [4]. Other identification methods have been developed for both linear and nonlinear systems [5,6]. However, current nonlinear identification techniques are only developed for systems with a small class of nonlinearities, and nonlinear system identification is currently an active field of research [3].

This paper describes a method for using the proper orthogonal decomposition (POD) of measured response data to construct a reduced-order model for predicting the free response of linear and nonlinear systems without any knowledge of the equations of motion for the structure. The POD is a statistical method for extracting significant shapes and their corresponding amplitude modulations that are present in a displacement-field history [7]. The POD is an attractive tool because it is a linear procedure and its governing mathematics are therefore relatively straightforward. However, it is often applied to nonlinear problems to compute the "optimal approximating *linear* manifold for the data" [7]. Although using a linear manifold to approximate a nonlinear system necessarily introduces error into an analysis, the POD does "not do the physical violence of linearization methods" [8]. The methods given in this paper use the POD to create a linear time-varying model for a nonlinear system that is more accurate than the model that would be obtained by linearizing the governing equations of motion.

The POD, also known as the Karhunen–Loève decomposition or principle component analysis, has been applied in many fields, including fluid mechanics, statistics, oceanography, meteorology, psychology, and economics. The method has been used in fluid mechanics to extract coherent structures from a turbulent flow [8] or to generate a basis for model reduction of unsteady viscous flows [9]. The POD has also recently been applied in structural dynamics for model order reduction [10–15], vibration control [16,17], structural health monitoring [18–20], modal analysis, sensor validation [21,22], nonlinear vibration analysis [23,24], and system identification [25–28].

The methods described in this paper add to previous applications of the POD for system identification. Many of the techniques that use the POD for system identification combine the model-reduction capabilities of the POD modes with standard system identification techniques for identification of reduced-order systems [25,26]. Other publications address model updating using the POD [27], and some estimate the degree of nonlinearity present in a system using the POD

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[28]. Currently, however, there is no technique that uses the information in the POD to construct a model for a system independently of standard identification methods. In this paper, a new method based on mode-summation theory is described for using data from the POD to simulate the free response of both linear and nonlinear systems without requiring knowledge of the equations of motion for the system (e.g., a finite element model).

This manuscript is organized into four sections. The first section describes how the POD may be calculated from a response (measured or simulated) using the singular value decomposition and explains the significance of each element of the POD. The second section describes methods for using the POD to predict the free response of a structure to initial displacements or velocities. The third section applies these methods to finite element simulations of two example problems: a linear cantilever beam and a damped beam with a cubic spring. The predicted responses are compared with those obtained from finite element models for each beam. The final section draws conclusions about the POD-based response-prediction methods based on the example problem results.

Computation of the POD

The POD can be computed by many methods [7]. This section explains how the POD is computed with the singular value decomposition of a snapshot matrix. First, a system response is generated by either forcing the system or imposing an initial velocity or displacement profile. In this paper, we will assume that an initial velocity profile $\dot{\mathbf{w}}_0$ or an initial displacement profile $\mathbf{w}_0$ is used to generate a response (not both at the same time). Next, the displacement at m degrees of freedom is sampled n times and the data are arranged in a "snapshot" matrix W:

$$W = \begin{bmatrix} w_1(t_1) & w_1(t_2) & \dots & w_1(t_n) \\ w_2(t_1) & w_2(t_2) & & \\ \vdots & & \ddots & \\ w_m(t_1) & & & w_m(t_n) \end{bmatrix} \tag{1}$$

Next, the singular value decomposition of W is computed:

$$W = U\Sigma V^T \tag{2}$$

In Eq. (2), the columns $\mathbf{u}_i$ of U are the proper orthogonal modes (POMs), the columns $\mathbf{v}_i$ of V are the proper orthogonal coordinate (POC) histories that correspond to each POM, and Σ is a diagonal matrix for which the diagonal elements σ_i are the proper orthogonal values (POVs) corresponding to each POM. The POC histories describe the amplitude modulation of each POM, and the POVs describe the relative significance of each POM in the response W [7]. If the system is linear and lightly damped, with a mass matrix proportional to the identity matrix, then the POMs will be equal to the linear normal modes. For nonlinear systems, if a single nonlinear normal mode is excited, then the first POM is a linear approximation to the excited nonlinear normal mode [29,30]. The percentage of signal energy captured by $\mathbf{u}_i$ is given by

$$\varepsilon_i = \frac{\sigma_i}{\sum_{j=1}^{m} \sigma_j} \tag{3}$$

Typically, only POMs that constitute a certain percentage of signal energy (e.g., 99 or 99.9%) are considered [7,8,26,31]. If k dominant POMs are considered, then we may approximate W as a summation of POMs and corresponding POC histories, shown next (noting that Σ is diagonal):

$$W \approx \sum_{i=1}^{k} \sigma_i \mathbf{u}_i \mathbf{v}_i^T \tag{4}$$

We note that even signals generated by nonlinear systems may be represented by a summation of POMs [3]. It should be noted that the POMs and POC histories are orthonormal:

$$U^T U = V^T V = I \tag{5}$$

Although this paper has focused on calculation of the POMs, POVs, and POC histories by performing a singular value decomposition, these quantities may also be determined when calculating the POD by other methods [7]. The singular value decomposition is used in this paper for its simplicity and convenient expression as a summation of modes.

Most applications of the POD use only the POMs, meaning that only the spatial information about the system is obtained. The only documented use for the POC histories is to examine their frequency content to determine which linear normal modes are represented in each POM [7,31]. This paper describes a new application for the POC histories by using them to form a reduced-order model for the free response of a system to the same type (i.e., displacement or velocity) of initial condition used to generate the original response.

Free-Response Prediction

This section provides a method for using the POMs and POC histories obtained from a response to predict the response to other initial conditions. Specifically, if an initial displacement is used to generate the original response, we give a method for predicting the response to a new initial displacement, and likewise for velocities. As shown in Eq. (4), the matrices U, V, and Σ completely describe a system's response to $\mathbf{w}_0$ or $\dot{\mathbf{w}}_0$ without requiring information about the full-order equations of motion. Suppose that an initial displacement profile $\mathbf{w}_0$ was imposed to form W. We now wish to modify the matrices to describe the system's response to a different initial displacement profile $\tilde{\mathbf{w}}_0$. When solving vibration problems (e.g., the wave equation) analytically using separation of variables, a typical approach is to express the response as a summation of spatial eigenfunctions, temporal functions, and coefficients that indicate the relative significance of each mode in the response. Although the eigenfunctions and temporal functions do not depend on the initial conditions, the significance coefficients do depend on them and are calculated using inner products of the eigenfunctions with the initial displacement or velocity profile [32]. We will take a similar approach and assume that U and V do not change for a given system, but that the participation of each POM, measured by σ_i, changes to represent the response to new initial conditions. If these assumptions are made, then the response $\tilde{W}$ to $\tilde{\mathbf{w}}_0$ may be written as

$$\tilde{W} \approx \sum_{i=1}^{k} \tilde{\sigma}_i \mathbf{u}_i \mathbf{v}_i^T \tag{6}$$

where the tilde notation indicates that the values in the diagonal matrix Σ have changed, although $\tilde{\Sigma}$ is still diagonal. The first column of Eq. (6), which corresponds to the initial time $t = t_0$, is

$$\tilde{\mathbf{w}}_0 \approx \sum_{i=1}^{k} \tilde{\sigma}_i \mathbf{u}_i v_{i,0} \tag{7}$$

In Eq. (7), the scalar $v_{i,0}$ is the first element in each POC history $\mathbf{v}_i$. We recall that the POMs are orthonormal and multiply both sides of Eq. (7) on the left by $\mathbf{u}_j^T$ to all but the jth term in the summation. The resulting equation may be solved for $\tilde{\sigma}_j$:

$$\tilde{\sigma}_j = \frac{\mathbf{u}_j^T \tilde{\mathbf{w}}_0}{v_{j,0}}, \qquad j = 1, 2, \dots, k \tag{8}$$

We note that $v_{j,0}$ in Eq. (8) is nonzero because the original POVs are found from

$$\sigma_j = \frac{\mathbf{u}_j^T \mathbf{w}_0}{v_{j,0}}, \qquad j = 1, 2, \dots, k \tag{9}$$

and are finite. After the new POVs $\tilde{\sigma}_j$ have been calculated, the response to $\tilde{\mathbf{w}}_0$ may be approximated using Eq. (6).

If an initial velocity profile $\dot{\mathbf{w}}_0$ is applied to form W, the response to a different velocity profile $\dot{\tilde{\mathbf{w}}}_0$ may be calculated in a very similar manner. Instead of calculating the new POVs from Eq. (8), however, we calculate them as follows:

$$\tilde{\sigma}_j = \frac{\mathbf{u}_j^T \dot{\tilde{\mathbf{w}}}_0}{\dot{v}_{j,0}}, \qquad j = 1, 2, \ldots, k \tag{10}$$

In Eq. (10), the time derivative of the POC histories at $t = t_0$ may be calculated from a forward-difference scheme or from the original velocity profile $\dot{\mathbf{w}}_0$, if it is known. At $t = t_0$, we may write

$$\dot{\mathbf{w}}_0 \approx \sum_{i=1}^{k} \sigma_i \mathbf{u}_i \dot{v}_{i,0} \tag{11}$$

Multiplying the left side of Eq. (11) by $\mathbf{u}_j^T$ allows us to solve for $\dot{v}_{j,0}$:

$$\dot{v}_{j,0} = \frac{\mathbf{u}_j^T \dot{\mathbf{w}}_0}{\sigma_j}, \qquad j = 1, 2, \ldots, k \tag{12}$$

By using the new POVs calculated from Eqs. (8) or (10) in Eq. (6), we are able to predict the free response of a system to a variety of initial conditions using only data obtained from the original POD.

Errors may be present in predictions made with the reduced-order model. First, the model can only accurately predict a response matrix that may be spanned by the original set of POMs. If initial conditions are introduced that generate a response that cannot be expressed by the POMs $\mathbf{u}_i$, then the prediction will be inaccurate. An extreme example arises when a model constructed from a response that is restricted to one plane is used to predict the response to an initial condition that excites motion that is restricted to a perpendicular plane. The measured POMs are entirely unable to represent the responses to these new initial conditions and the method predicts no response at all. Therefore, when constructing the model, it is desirable to use a response to initial conditions that excite a wide variety of shapes to generate POMs that can represent responses to a large selection of initial conditions. When the method is applied to nonlinear systems, a linear model is obtained that will match the measured nonlinear response exactly. For nonlinear systems, a new excitation can result in a significant change in the natural frequencies of the system and cause the system to exhibit new behaviors [33]. The reduced-order model constructed from the POD will be unable to predict these new effects. However, the model may accurately predict the response to an initial condition that is nearby the original condition.

Examples

This section applies the methods described in the previous section to two models: an undamped cantilever beam and the same beam with a dashpot and cubic spring at the end, shown in Fig. 1. The beams were steel ($E = 200$ GPa and $\rho = 8000$ kg/m^3) and had dimensions of $24 \times 1 \times 0.75$ in. Finite element models were created for each beam using 24 beam elements. The spring force over the tip-displacement range seen in simulations in this paper is shown in Fig. 2, along with the cubic-force equation, and the value of the dashpot was 0.2 lbf $\cdot$ s/in.

The Newmark method was used to simulate the exact response of each beam model to several initial displacement profiles, shown in Fig. 3. Each displacement profile was generated by applying static loads to the linear beam. The displacement profile $\mathbf{w}_0$ was generated by applying static loads of 300, 200, and −260 lbf at locations 17, 18, and 24 in. from the root of the beam. Profiles $\mathbf{w}_1$ and $\mathbf{w}_2$ were formed by applying static loads of 100 and −300 lbf at locations 24 and 8 in. from the root of the beam, respectively. The free responses of each model to each initial displacement were simulated for 0.05 s (approximately twice the fundamental period of 0.0238 s for the linear beam), and the vertical displacements at 24 points were captured at every time step (0.1 ms) to form W. Thus, the dimensions of W were (24×500) for both models.

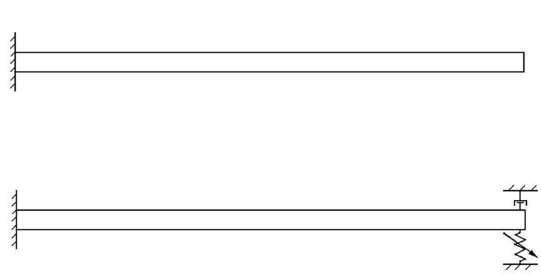

Fig. 1 Linear (top) and nonlinear (bottom) beam models.

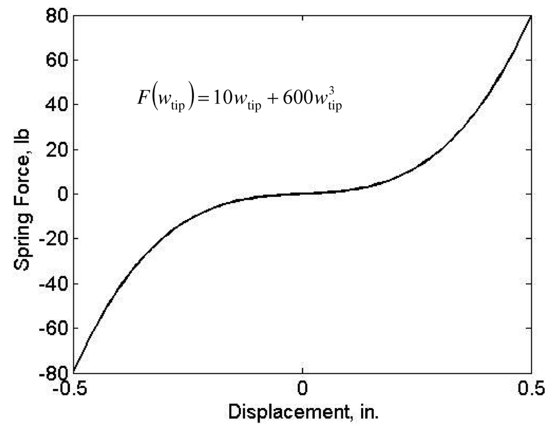

Fig. 2 Cubic spring force for the nonlinear beam model.

Next, the POD was computed from each beam's response to $\mathbf{w}_0$ and the methods described in Eqs. (6–9) were applied to simulate the responses of both systems to initial displacement profiles $\mathbf{w}_1$ and $\mathbf{w}_2$. The responses were simulated using the first four POMs, which corresponded to 99.9% of the signal energy. The original POVs (from the response to $\mathbf{w}_0$) and recalculated POVs (for responses to $\mathbf{w}_1$ and $\mathbf{w}_2$) are shown in Fig. 4 for the linear beam.

The original and recalculated POVs for the nonlinear beam were very similar and are not shown. In both cases, the POVs drop off very quickly, showing that only a few POMs are necessary to represent the motion accurately. This figure also illustrates how the significance of each mode changes in response to the different initial displacement profiles. For example, the first POV for both beams is larger in response to $\mathbf{w}_1$ than to $\mathbf{w}_0$, indicating that the first POM is more active in the response to $\mathbf{w}_1$ than to $\mathbf{w}_0$. For visualization purposes, the first three POMs for the linear beam are shown in Fig. 5 (the POMs for the nonlinear beam are very similar).

The tip displacements at each time step calculated by the FE model and the POD-based model for the linear beam responses to $\mathbf{w}_1$ and $\mathbf{w}_2$ are shown in Figs. 6 and 7. Both figures show that the POD-based model predicts the tip displacement very accurately for the linear beam. The tip displacements for the nonlinear beam responses to $\mathbf{w}_1$ and $\mathbf{w}_2$ are shown in Figs. 8 and 9. In each figure, the results obtained

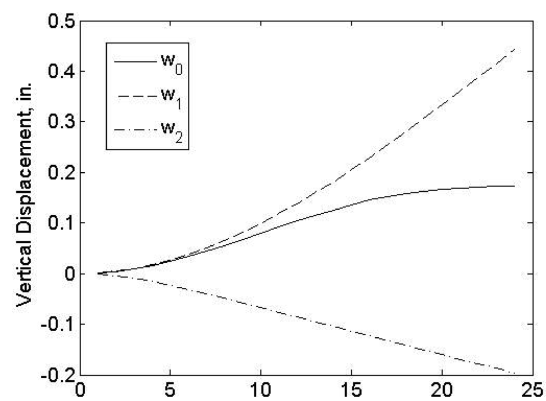

Fig. 3 Applied initial displacement profiles.

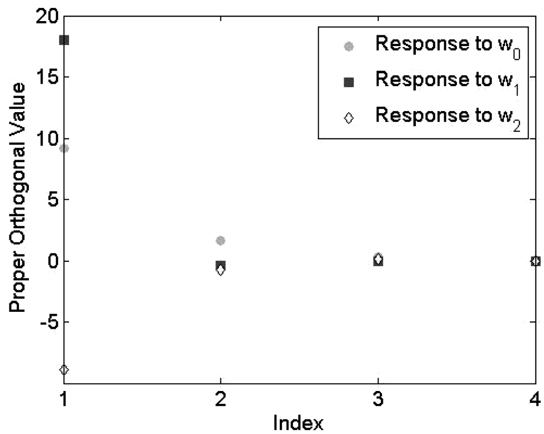

Fig. 4 Original and recalculated POVS for the linear beam model.

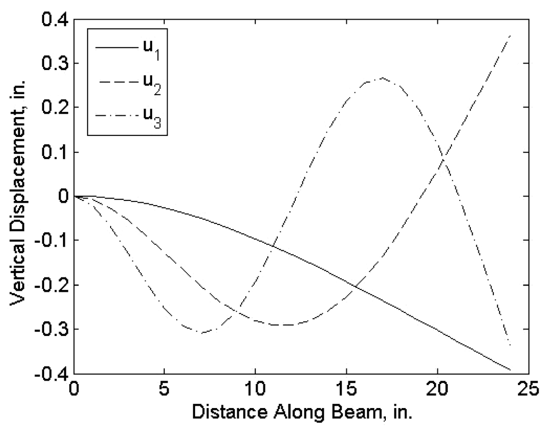

Fig. 5 First three POMS for the linear beam model.

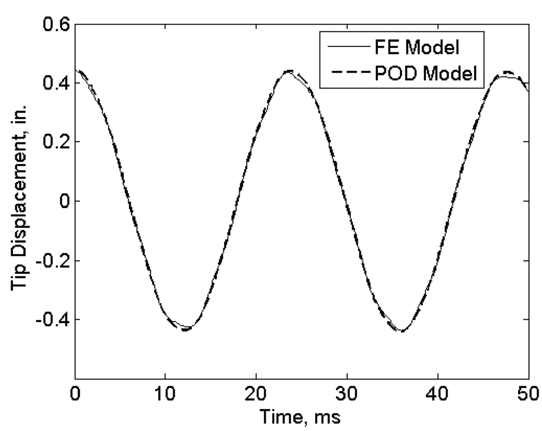

Fig. 6 Tip displacement of the linear beam model in response to $\mathbf{w}_1$.

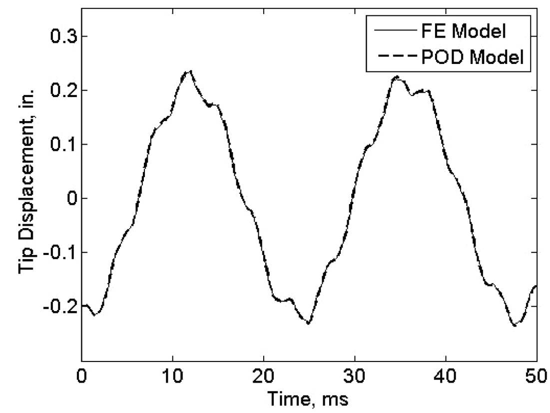

Fig. 7 Tip displacement of the linear beam model in response to $\mathbf{w}_2$.

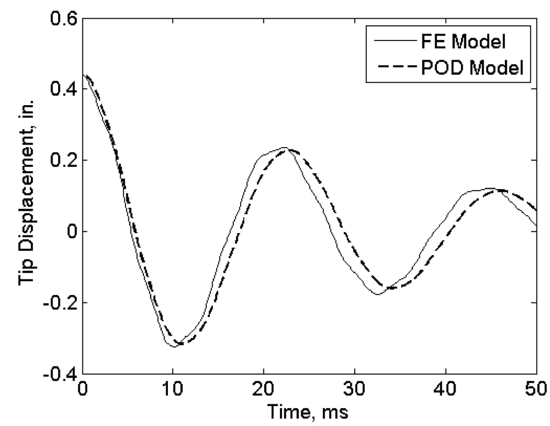

Fig. 8 Tip displacement of the nonlinear beam model in response to $\mathbf{w}_1$.

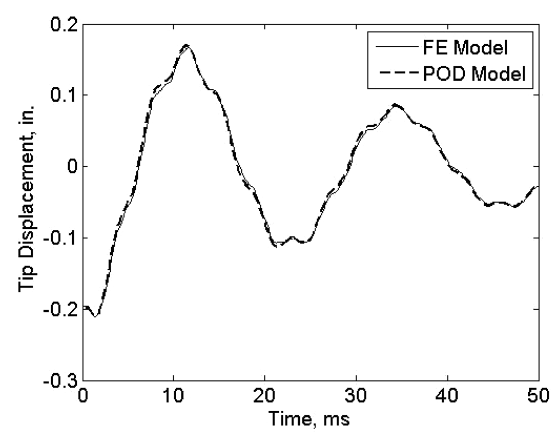

Fig. 9 Tip displacement of the nonlinear beam model in response to $\mathbf{w}_2$.

using the POD-based models are plotted with the exact results obtained by the FE models. The amplitude of the tip displacement decreases with time, correctly modeling the damping, but a frequency error is visible in the prediction. This error is due to the fact that the natural frequencies of the nonlinear beam have changed with the new initial condition and, as mentioned previously, the POD-based model is linear with a fixed frequency equal to the frequency observed in the response to $\mathbf{w}_0$.

Next, we applied the methods described in Eqs. (6–12) to the same two beam models (Fig. 1) to predict the initial velocity response of each beam. As with the initial displacement example, FE models for each beam were used to simulate the exact response of each beam to several initial velocity profiles, shown in Fig. 10. The initial velocity profile $\dot{\mathbf{w}}_0$ was generated by applying a 1000-lbf impulse at the beam tip on the initial time step and measuring the velocity at the second time step. Similarly, the initial velocity profiles $\dot{\mathbf{w}}_1$ and $\dot{\mathbf{w}}_2$ were generated from a 1000-lbf impulse 2 in. from the beam root and 5000- and 3000-lbf impulses 3 and 23 in. from the beam root, respectively. Nine POMs and POC histories (capturing 99.9% of the original signal energy) were obtained from the response of each model to the initial velocity profile $\dot{\mathbf{w}}_0$ and used to predict the responses to $\dot{\mathbf{w}}_1$ and $\dot{\mathbf{w}}_2$ for each beam.

The tip displacements in response to $\dot{\mathbf{w}}_1$ and $\dot{\mathbf{w}}_2$ predicted by the POD-based model are plotted with the FE model results in Figs. 11 and 12 for the linear beam and in Figs. 13 and 14 for the nonlinear beam. Again, the POD-based model accurately predicted the tip displacement for the linear beam. The POD-based model for the nonlinear beam is also a good approximation to the nonlinear model

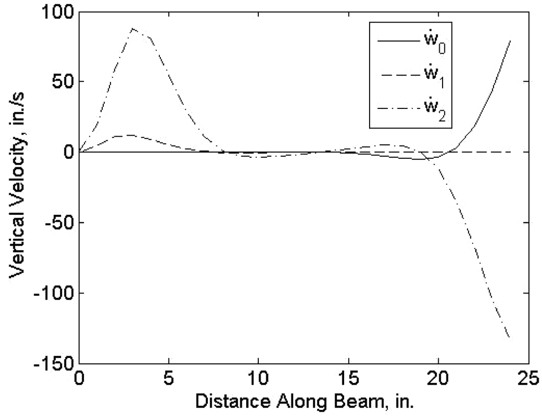

Fig. 10 Applied initial velocity profiles.

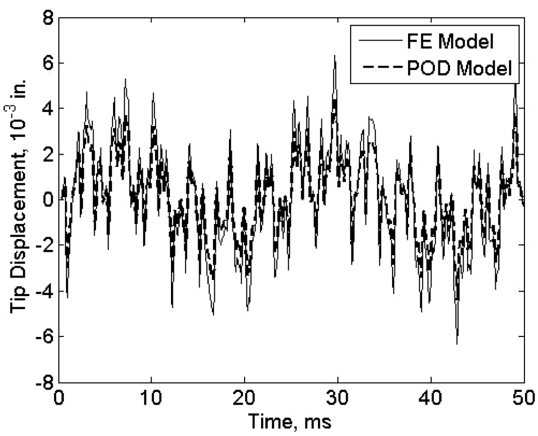

Fig. 11 Tip displacement of the linear beam model in response to $\dot{w}_1$.

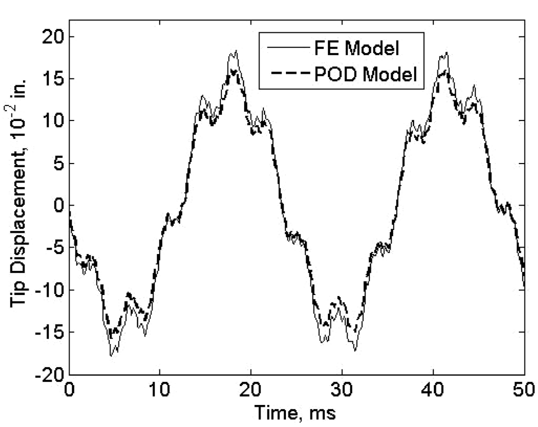

Fig. 12 Tip displacement of the linear beam model in response to $\dot{w}_2$.

for the initial velocity responses shown, but as stated before, incorrect predictions will occur if the nonlinear system undergoes frequency changes and/or exhibits other nonlinear characteristics not observed in the original signal.

Conclusions

Based on the results shown in the paper, we conclude that the method of proper orthogonal value recalculation explained in this research can be used to construct an accurate reduced-order model for the free response of linear and nonlinear structural systems without knowledge of the equations of motion for the structure. For both linear and nonlinear systems, the accuracy of the prediction will depend on how well the original set of proper orthogonal modes can represent the new response. For nonlinear systems, the method can

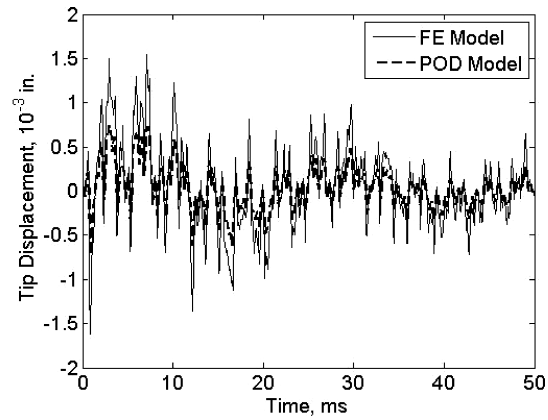

Fig. 13 Tip displacement of the nonlinear beam model in response to $\dot{w}_1$.

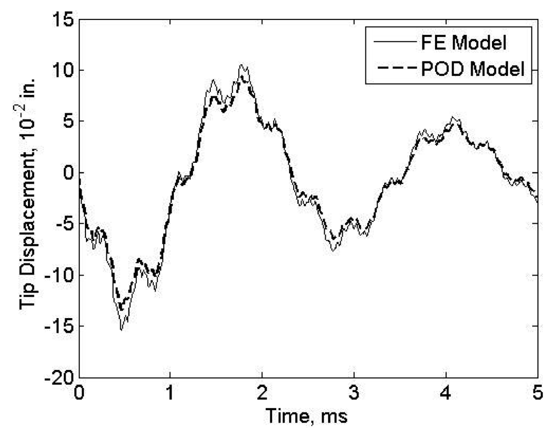

Fig. 14 Tip displacement of the nonlinear beam model in response to $\dot{w}_2$.

be used to replicate the original nonlinear response accurately, but the method is linear and can only predict responses to new initial conditions accurately if they exhibit the same frequency content and nonlinear characteristics that are present in the original response.

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